
Equivalence Checking of Quantum Circuits

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April 8, 2025

DQC Scientific Quantum Conference



DEPARTMENT
OF COMPUTER
SCIENCE

This talk is based on joint work¹

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Simon Brun Olsen



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VILLUM FONDEN



¹C. B. Larsen, S. B. Olsen, K. G. Larsen, and C. Schilling. “Contraction heuristics for tensor decision diagrams”. *Entropy* (2024).

Overview

Equivalence checking of quantum circuits

Tensor networks and tensor decision diagrams

Equivalence checking based on tensor decision diagrams

Empirical evaluation

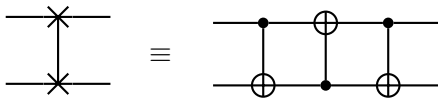
Conclusion and future work

Quantum circuit compilation

- When designing quantum algorithms, it is useful to have many types of operations available
- Real quantum computers only support a few types of operations
- A **compiler** translates a high-level circuit with many gate types to a low-level circuit with few gate types
- Important that the compiled circuits are **equivalent** (i.e., compute the same output for the same input)

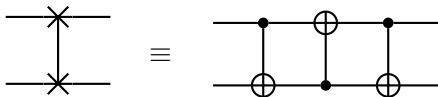
How to implement a SWAP gate?

- The SWAP gate is **equivalent** to three CNOT gates



How to implement a SWAP gate?

- The SWAP gate is **equivalent** to three CNOT gates



- We can easily prove this by comparing the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Algorithm to check equivalence of quantum circuits

- Given: Two circuits C_1, C_2
- Question: Are the circuits equivalent ($C_1 \equiv C_2$)?

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 1. Compute matrix representations U_1, U_2
 2. Check equality up to a factor (global phase $e^{i\theta}$)

$$U_1 \stackrel{?}{=} e^{i\theta} \cdot U_2$$

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- Problem: Matrices are **exponentially large**
For n qubits $\rightsquigarrow 2^n \times 2^n$

Equivalence checking is hard

- Checking exact equivalence is co-NQP-complete¹
- Checking approximate equivalence is co-QMA-complete^{2,3}
- Problems in these complexity classes are widely believed to require exponential computations in the worst case

¹Y. Tanaka. “Exact non-identity check is NQP-complete”. *Int. J. Quantum Inf.* (2010).

²D. Janzing, P. Wocjan, and T. Beth. ““Non-identity-check” is QMA-complete”. *Int. J. Quantum Inf.* (2005).

³Z. Ji and X. Wu. *Non-identity check remains QMA-complete for short circuits*. 2009. arXiv: 0906.5416.

Approaches to equivalence checking

- ZX-calculus¹
- Encoding with decision diagrams² ← relevant later
- Tensor network contraction³ ← relevant later
- Simulation-based approach for the Clifford group⁴
- Weighted model counting⁵

¹T. Peham, L. Burgholzer, and R. Wille. “Equivalence checking of quantum circuits with the ZX-calculus”. *IEEE J. Emerg. Sel. Topics Circuits Syst.* (2022).

²L. Burgholzer and R. Wille. “Advanced equivalence checking for quantum circuits”. *IEEE Trans. Comput. Aided Des. Integr. Circuits Syst.* (2021).

³R. Orús. “Tensor networks for complex quantum systems”. *Nature Reviews Physics* (2019).

⁴D. Thanos, T. Coopmans, and A. Laarman. “Fast equivalence checking of quantum circuits of Clifford gates”. *ATVA*. 2023.

⁵J. Mei, T. Coopmans, M. M. Bonsangue, and A. Laarman. “Equivalence checking of quantum circuits by model counting”. *IJCAR*. 2024.

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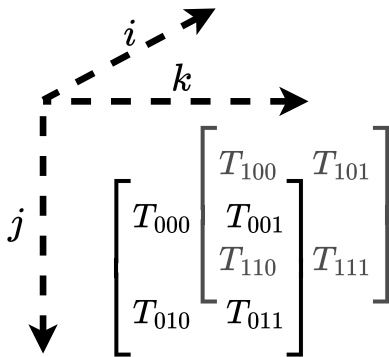
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Tensors

- Generalization of vectors / matrices to higher dimensions



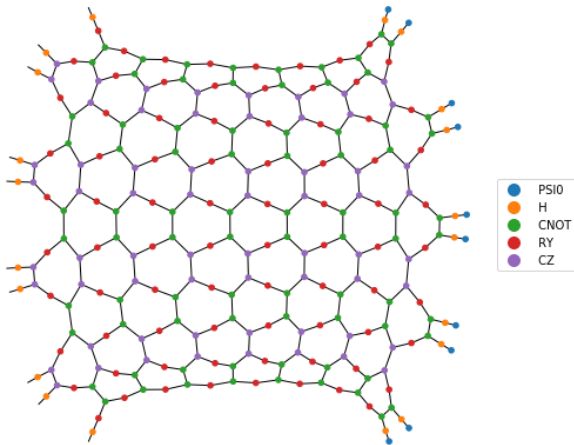
Tensors

- Generalization of vectors / matrices to higher dimensions
- High-level graphical representation as a node with edges

vector	v_j	
matrix	M_{ij}	
3-index tensor	T_{ijk}	

Tensor networks

- Tensors can be arranged in a graph



Tensor networks

- Tensors can be arranged in a graph
- Nodes with shared edges can be **contracted** (= merged)

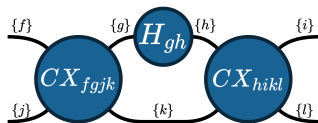
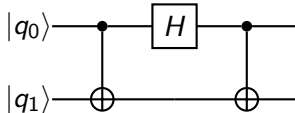
Corresponds to matrix-vector and matrix-matrix multiplication for special cases

$$\text{---} \bigcirc_i \text{---} \bigcirc_j \text{---} = \sum_j A_{ij} \underbrace{v_j}$$

$$\text{---} \bigcirc_i \text{---} \bigcirc_j \text{---} \bigcirc_k \text{---} = \sum_j A_{ij} \underbrace{B_{jk}} = AB$$

Example

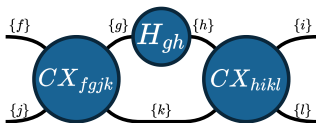
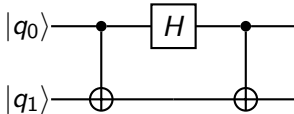
1. Initial tensor network has one tensor for each gate
Three choices for contraction ($\{g\}$, $\{h\}$, $\{k\}$)



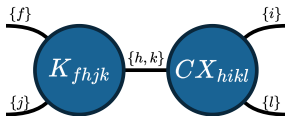
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Example

- Initial tensor network has one tensor for each gate
Three choices for contraction ($\{g\}$, $\{h\}$, $\{k\}$)
- Contraction of CX_{fgjk} and H_{gh} via $\{g\}$



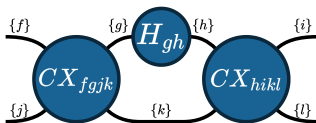
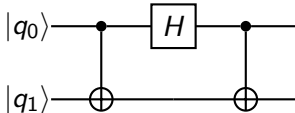
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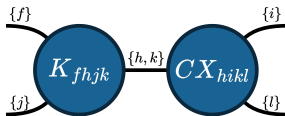
2.

Example

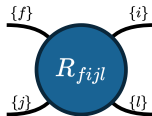
1. Initial tensor network has one tensor for each gate
Three choices for contraction ($\{g\}$, $\{h\}$, $\{k\}$)
2. Contraction of CX_{fgjk} and H_{gh} via $\{g\}$
3. Contraction of remaining two tensors



1.



2.



3.

Application: Quantum simulation on classical computer²

- Contract tensors in smart orders
- Different contraction heuristics to minimize floating-point operations, size, etc.¹

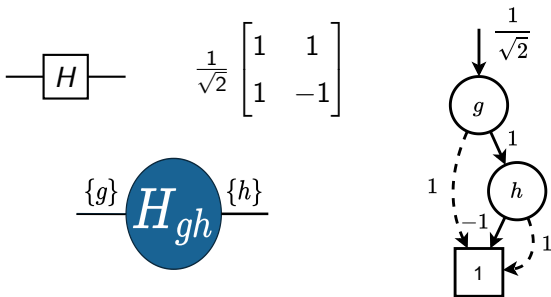
¹J. Gray and S. Kourtis. “Hyper-optimized tensor network contraction”. *Quantum* (2021).

²I. L. Markov and Y. Shi. “Simulating quantum computation by contracting tensor networks”. *SIAM J. Comput.* (2008).

Tensor decision diagrams (TDDs)

- Alternative, unique representation of a tensor
- Informal introduction by example

Example: Hadamard gate with matrix, tensor, and TDD



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Alternative “reverse scheme” for equivalence¹

$$C_1 \equiv C_2 \stackrel{\text{def}}{\iff} \exists \theta: U_1 = e^{i\theta} \cdot U_2$$

$$\iff \exists \theta: U_1 \cdot U_2^\dagger = e^{i\theta} \cdot I \stackrel{\text{def}}{\iff} C_1 C_2^{-1} \equiv C_I$$

- C_2^{-1} is the inverted C_2 (reversed and each gate inverted)
- Allows to combine both circuits

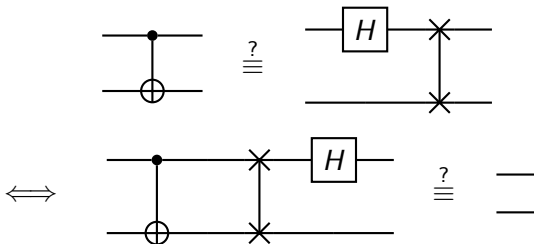
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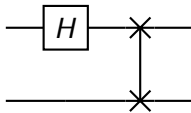
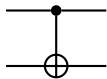
- (Coincidentally, the swap and Hadamard gates are self-inverse)

¹G. F. Viamontes, I. L. Markov, and J. P. Hayes. “Checking equivalence of quantum circuits and states”. *ICCAD*. 2007.

New algorithm for equivalence checking

Algorithm combines reverse scheme,
tensor networks, and TDDs

Given: Quantum circuits C_1 , C_2

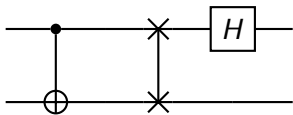
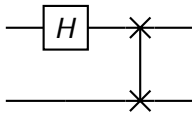
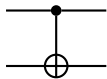


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1. Construct circuit $C_1 C_2^{-1}$

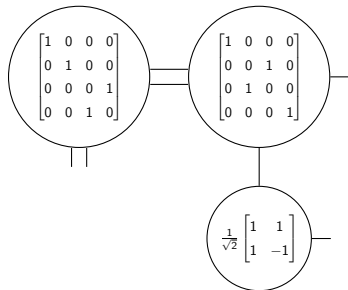
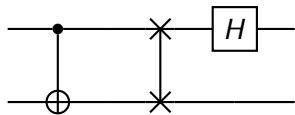


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Algorithm combines reverse scheme,
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Given: Quantum circuits C_1, C_2

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2. Convert $C_1 C_2^{-1}$ to tensor network

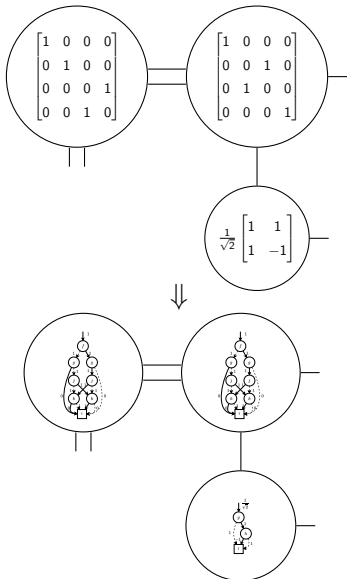


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Given: Quantum circuits C_1, C_2

1. Construct circuit $C_1 C_2^{-1}$
2. Convert $C_1 C_2^{-1}$ to tensor network
3. Convert all tensors to TDDs



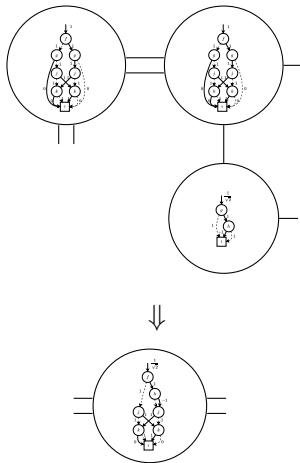
(TDDs on the right are only exemplary)

New algorithm for equivalence checking

Algorithm combines reverse scheme,
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Given: Quantum circuits C_1, C_2

1. Construct circuit $C_1 C_2^{-1}$
2. Convert $C_1 C_2^{-1}$ to tensor network
3. Convert all tensors to TDDs
4. Contract TDD network



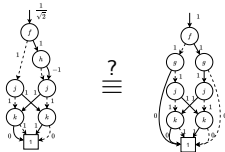
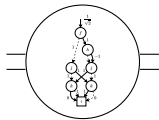
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1. Construct circuit $C_1 C_2^{-1}$
2. Convert $C_1 C_2^{-1}$ to tensor network
3. Convert all tensors to TDDs
4. Contract TDD network
5. Compare TDD to identity TDD



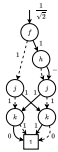
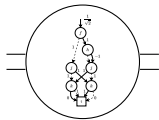
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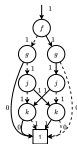
Algorithm combines reverse scheme,
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Given: Quantum circuits C_1, C_2

1. Construct circuit $C_1 C_2^{-1}$
2. Convert $C_1 C_2^{-1}$ to tensor network
3. Convert all tensors to TDDs
4. **Contract TDD network** ← how?
5. Compare TDD to identity TDD



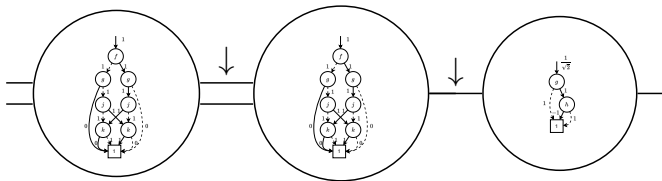
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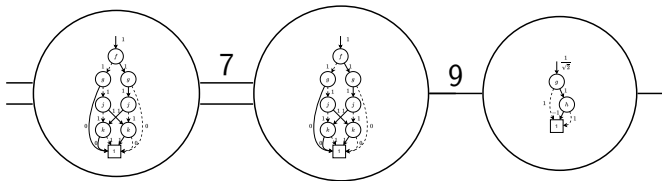
Lookahead heuristic for TDD contraction

- Greedy algorithm (finds a local optimum)
- In each contraction step:
 1. Evaluate all possible contractions



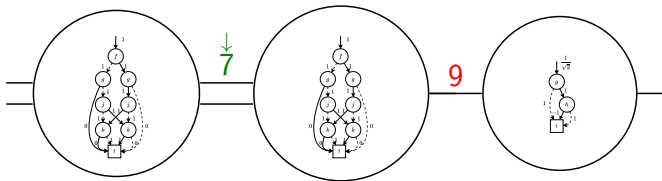
Lookahead heuristic for TDD contraction

- Greedy algorithm (finds a local optimum)
- In each contraction step:
 1. Evaluate all possible contractions
 2. Measure size of resulting TDDs



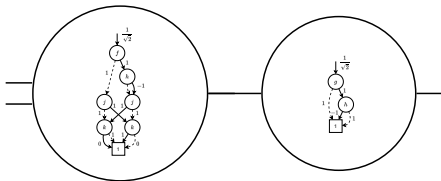
Lookahead heuristic for TDD contraction

- Greedy algorithm (finds a local optimum)
- In each contraction step:
 1. Evaluate all possible contractions
 2. Measure size of resulting TDDs
 3. Execute a contraction with smallest output



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Lookahead heuristic for TDD contraction

- Greedy algorithm (finds a local optimum)
- In each contraction step:
 1. Evaluate all possible contractions
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 3. Execute a contraction with smallest output
- Why should this scale?
 - Network is sparsely connected
 - Initial contractions (with many tensors) are cheap
 - Results can be stored for later iterations
- Still, step 1. is quite expensive

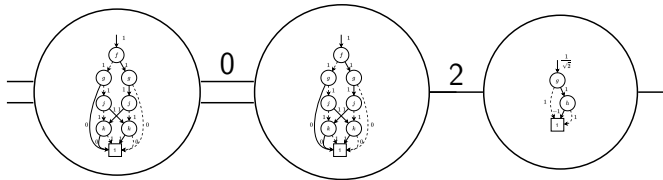
Counting heuristic for TDD contraction

- Goal: Imitate lookahead heuristic without expensive step 1
- Empirical observation: Lookahead heuristic prefers to distribute the contractions over the tensor network

¹We set one edge to “2” to get an interesting example

Counting heuristic for TDD contraction

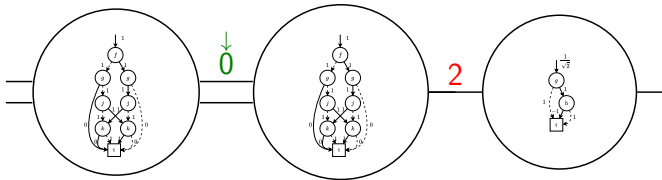
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Counting heuristic for TDD contraction

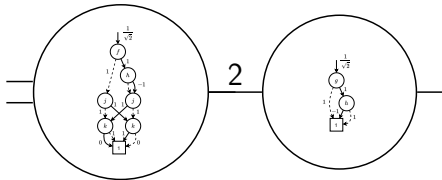
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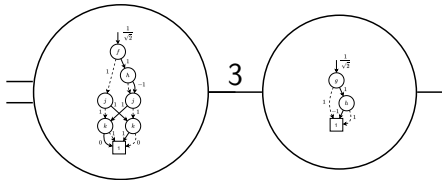
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Counting heuristic for TDD contraction

- Goal: Imitate lookahead heuristic without expensive step 1
- Empirical observation: Lookahead heuristic prefers to distribute the contractions over the tensor network
- In each contraction step:
 1. Select nodes with oldest participation in contraction¹
 2. Update usage statistics of neighboring edges



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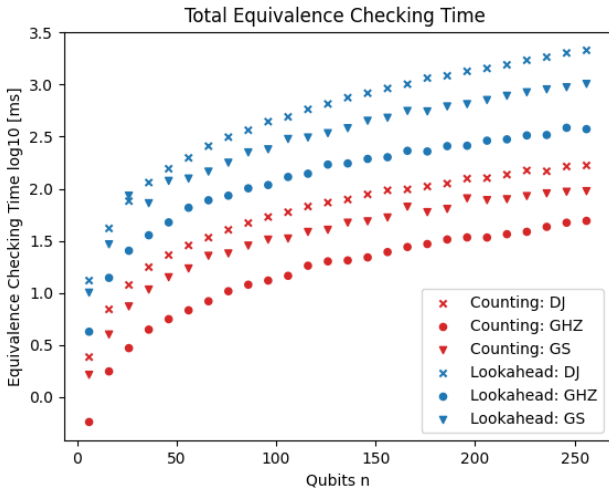
Conclusion and future work

Quantum circuits in evaluation

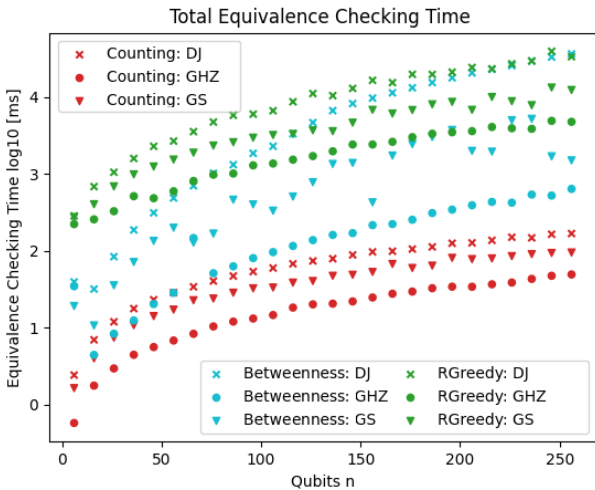
- Circuits from MQT Bench¹ with varying number of qubits at two compilation levels (level 1 and 3 (out of 4)) with significantly different gate sets and layouts
 - Deutsch-Jozsa algorithm (DJ)
 - Greenberger-Horne-Zeilinger state preparation (GHZ)
 - Graph state preparation (GS)

¹N. Quetschlich, L. Burgholzer, and R. Wille. “MQT Bench: Benchmarking software and design automation tools for quantum computing”. *Quantum* (2023).

Comparison of own heuristics

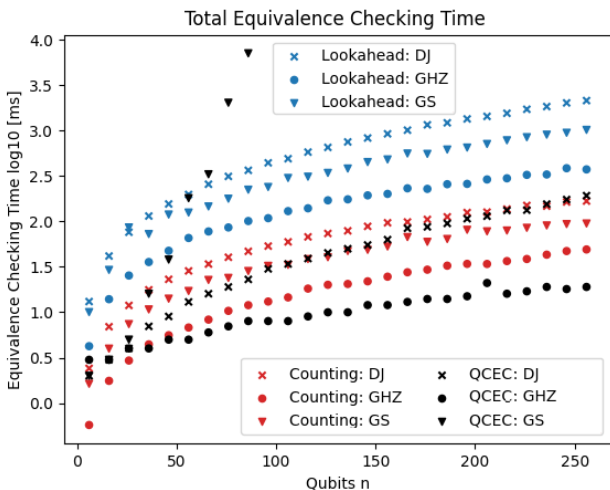


Comparison to cotengra¹



¹J. Gray and S. Kourtis. “Hyper-optimized tensor network contraction”. *Quantum* (2021).

Comparison to QCEC¹



¹L. Burgholzer and R. Wille. “QCEC: A JKQ tool for quantum circuit equivalence checking”. *Softw. Impacts* (2021).

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Conclusion

- Integration of “reverse scheme” and TDD networks
- Lookahead heuristic (greedy)
- Counting heuristic (cheap approximation)
- Evaluation:
 - Outperforms cotengra’s heuristics
 - Often keeps up with QCEC

Future work & FMQC 2025

- Exploit parallelization (CPU, GPU)
- Find other heuristics
 - Generalize tensor network heuristics to TDD networks
 - Identify equivalent subcomponents (modularity)
 - Employ machine learning
- New PhD project since December 2024



Workshop on Formal Methods in Quantum Computing
co-located with CONFEST 2025 in Aarhus, August 25

<https://fmqc-workshop.github.io/2025/>